Exercise 6

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = 2x - 2\sin x + \cos x - \int_0^x (x-t)^2 u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = 2x - 2\sin x + \cos x - \int_0^x (x-t)^2 \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 2x - 2\sin x + \cos x + (-2)\int_0^x \frac{(x-t)^2}{2}[u_0(t) + u_1(t) + \cdots]\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{2x - 2\sin x + \cos x}_{u_0(x)} + \underbrace{(-2)\int_0^x \frac{(x-t)^2}{2} u_0(t)\,dt}_{u_1(x)} + \underbrace{(-2)\int_0^x \frac{(x-t)^2}{2} u_1(t)\,dt}_{u_2(x)} + \cdots$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. Note that having $(x-t)^2/2$ in the integrand means we integrate the function next to it three times (from 0 to x).

$$u_0(x) = 2x - 2\sin x + \cos x$$

$$u_1(x) = -2\int_0^x \frac{(x-t)^2}{2} u_0(t)\,dt = -2\left[\frac{x^4}{12} - 2\left(\cos x + \frac{x^2}{2} - 1\right) + (x - \sin x)\right]$$

$$= -4 - 2x + 2x^2 - \frac{x^4}{6} + 4\cos x + 2\sin x$$

$$\vdots$$

The noise terms, $\pm 2x$ and $\mp 2\sin x$, appear in both $u_0(x)$ and $u_1(x)$. Cancelling $2x$ and $-2\sin x$ from $u_0(x)$ leaves $\cos x$. Now we check to see whether $u(x) = \cos x$ satisfies the integral equation.

$$\cos x \stackrel{?}{=} 2x - 2\sin x + \cos x - \int_0^x (x-t)^2 \cos t\,dt$$

$$\cos x \stackrel{?}{=} \cancel{2x} - \cancel{2\sin x} + \cos x - 2(\cancel{x} - \cancel{\sin x})$$

$$\cos x = \cos x$$

Therefore,

$$u(x) = \cos x.$$